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Coupling Local and Non-local Problems:

Alternating Schwarz and Optimization-based Approaches

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Introduction

We compare alternating Schwarz method with optimization-based approach for coupling nonlocal and local operators.

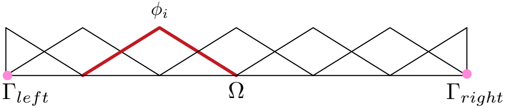
- Requirements: Octave or Matlab with optimization toolbox
- `git clone https://github.com/kyungjoo-kim/cm4.git your-local-directory`

Warm Up: Local and Nonlocal 1D Poisson Problem

For given f and θ, we seek solution u of two point boundary value problem:

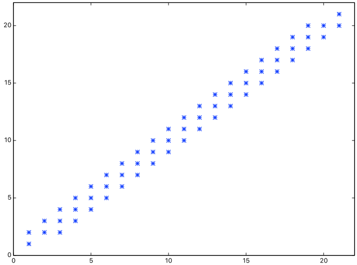
$$\begin{aligned} -\Delta u &= f \quad x \in \Omega, \\ u &= \theta \quad x \in \Gamma. \end{aligned}$$

Warm Up: Local and Nonlocal 1D Poisson Problem

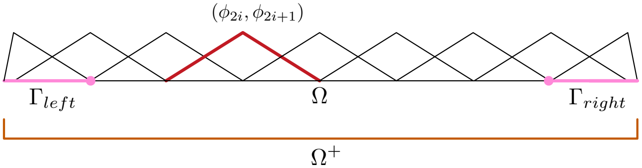


Weak form of 1D local diffusion model is

$$\int_\Omega \nabla u_l \nabla z_l \, dx = \int_\Omega f_l z_l \, dx.$$

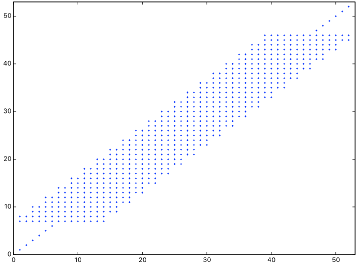


Warm Up: Local and Nonlocal 1D Poisson Problem



Weak form of 1D nonlocal diffusion model is

$$\int_{\Omega^+} \int_{\Omega^+ \cap B_\varepsilon(x)} \frac{1}{\varepsilon^2 |x-y|} \left(u_n(y) - u_n(x)\right)\left(z_n(y) - z_n(x)\right) dydx = \int_{\Omega} f_n(x) z_n \, dx.$$



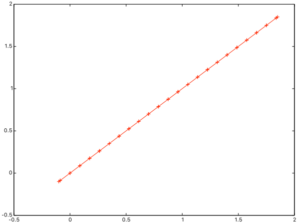
Exercise: Complexity of Local and Nonlocal Operators

Compare the local and nonlocal operators.

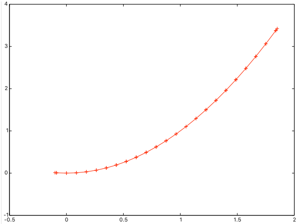
```
%% N : # of elements
%% epsilon : nonlocal interaction radius
%% test : test problem id, see exact_solution.m and source_integral.m

%% 1. local operator
run_local_problem(N, epsilon, test)

%% 2. nonlocal operator
run_nonlocal_problem(N, epsilon, test)
```

- How expensive is constructing the nonlocal operator (nnz and integration cost) ?

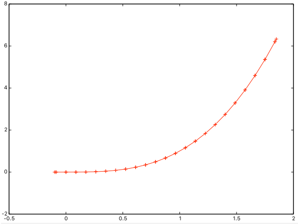
Manufactured Solution in Test Problems



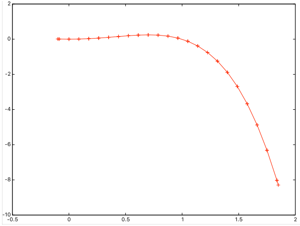
test 1: $u = x$



test 2: $u = x^2$

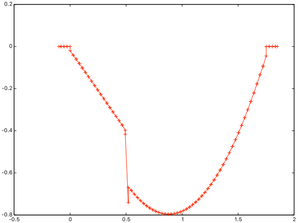


test 3: $u = x^3$



test 4: $u = x^2 - x^4$

Manufactured Solution in Test Problems



test 0: Solution with discontinuity

$$f_n = f_l = \begin{cases} 0 & x < \frac{1}{2} - \varepsilon \\ -\frac{2}{\varepsilon}\left(\frac{1}{2}\varepsilon^2 - \varepsilon + \frac{3}{8} + \left(2\varepsilon - \frac{3}{2} - \log\varepsilon\right)x + \right. \\ \qquad \left.\left(\frac{3}{2} + \log\varepsilon\right)x^2 - \log\left(\frac{1}{2} - x\right)\left(x^2 - x\right)\right) & \frac{1}{2} - \varepsilon \le x < \frac{1}{2} \\ -\frac{2}{\varepsilon}\left(\frac{1}{2}\varepsilon^2 - \varepsilon - \frac{3}{8} + \left(2\varepsilon + \frac{3}{2} + \log\varepsilon\right)x - \right. \\ \qquad \left.\left(\frac{3}{2} + \log\varepsilon\right)x^2 - \log\left(x - \frac{1}{2}\right)\left(x^2 - x\right)\right) & \frac{1}{2} \le x < \frac{1}{2} + \varepsilon \\ 0 & x \ge \frac{1}{2} + \varepsilon. \end{cases}$$

test 0: Source function

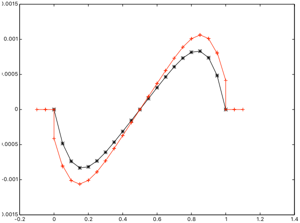
Comparison of Local and Nonlocal Operators

With decreasing epsilon, the nonlocal operator becomes close to the local operator

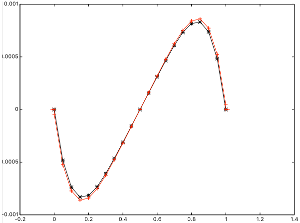
$$-Lu = -\Delta u + \varepsilon^2 D^4(u) + O(\varepsilon^4)$$

where $D^4(u)$ is a combination of the 4th derivatives of u.

```
    %% N : # of elements
    %% epsilon : nonlocal interaction radius
    %% niter : # of test runs
    run_comparison_local_nonlocal(N, epsilon, niter)
```

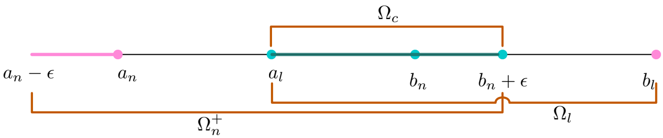


$\varepsilon = 0.1$



$\varepsilon = 0.1/16$

Exercise: Alternating Schwarz and Optimization-based Coupling



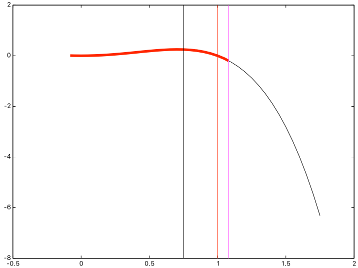

```
% NN — # of elements in nonlocal region
% NL — # of elements in local region
% epsilon — interaction radius
% test — problem id
run_coupling_alternate(NN, NL, epsilon, test)
run_coupling_optimize(NN, NL, epsilon, test)
```

- Confirm that the coupled solutions converge for test problems.
- Check the convergence for different overlapping regions.
- Try to use different mesh resolutions for a more realistic problem (fine mesh for nonlocal operator and coarse mesh for local operator).

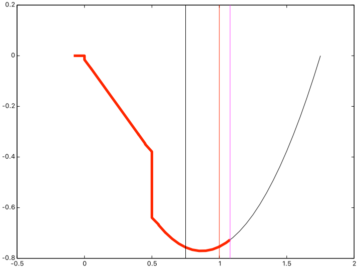
Examples of Glued Solution

Play with the code changing parameters:

```
    % problem domain setup
    problem_domain  = [ 0    1.75];
    nonlocal_domain = [ 0    1   ];
    local_domain    = [ 0.75 1.75];
```



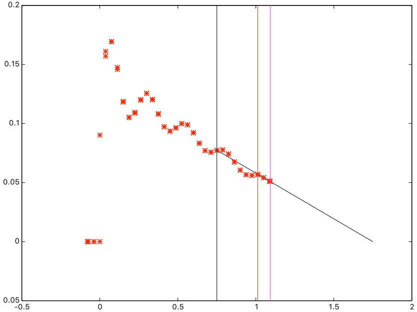
test 4: $u = x^2 - x^4$



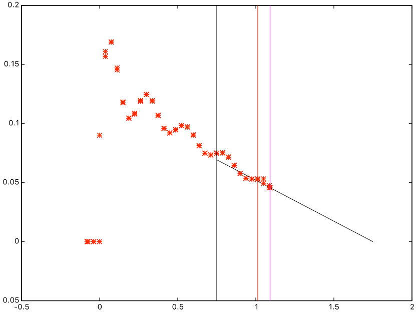
test 0: Solution with discontinuity

Exercise: Difference in Two Coupling Strategies

```
run_coupling_alternate(20,20,0.08,6)
run_coupling_optimize(20,20,0.08,6)
```



test 6: Alternate Schwarz coupling



test 6: Optimization-based coupling

Summary

- Both alternating Schwarz and optimization-based coupling strategies glue the solutions without modifying the original equations and properties.
- For more realistic problems with different physics models, the optimization-based coupling approach provides robust and unique solution.
- Optimization-based approach can provide application specific object function to define coupling mechanism.